

On a class of gauge theories

E.K. Loginov*

*Department of Physics, Ivanovo State University
Ermaka Street 39, Ivanovo 153025, Russia*

Abstract

We give a framework to describe gauge theory in which a nonassociative Moufang loop takes the place of the structure group. The structure of such gauge theory has many formal similarities with that of Yang-Mills theory. We extend the gauge invariance to this theory and construct an on-shell version of $N = 1$ supersymmetric gauge theory.

1 Introduction

In the past few ten years there have been many attempts to incorporate the unique algebra of octonions into physics. From the early 1970s and up to the present time, octonions have been applied with some success to different important problems such as quark confinement and grand unified model, see Ref. 1. Starting from the 1980s, new applications of octonions in physics were found, the instanton problem, supersymmetry, supergravity, superstrings, and recently branes technology. Application of octonions to supergravity spontaneous compactification was a very important and active field of research during the mid-1980s, especially compactification of $d = 11$ supergravity over S^7 to four dimensions. It is an impossible task to list all the relevant papers, so we direct the interested reader to Ref. 2, there a lot of references are given. We just mention that the first indication of the octonionic nature of this problem appeared in the Englert solution of $d = 11$

*E-mail address: ek.loginov@mail.ru

supergravity over \mathbb{S}^7 . The relation between superstrings (p branes) and octonions had been considered from many different points of view, the reader may consult the references given in Ref. 3 for details. Recently nonassociativity is known to appear in open string theory with nonconstant background $B_{\mu\nu}$ field, see Ref. 4. It was also argued that the algebra of closed string field theory should be commutative nonassociative. In Ref. 5, they discussed commutative nonassociative gauge theory with Lorentz and Poincaré symmetry. There also other discussions on nonassociative theory, see Ref. 6.

In the paper we attempt to construct a gauge theory based on the octonion algebra in familiar manner of Yang-Mills theory. The paper is organized as follows. In Section 2 we list the properties of octonions and some other mathematical structures relevant to our work. In Section 3 we extend the gauge invariance to the theory in which a nonassociative Moufang loop take the place of the structure group. In Section 4 we construct an on-shell version of $N = 1$ supersymmetric gauge theory without matter. Section 5 contain some concluding remarks. In order to make the paper self-consistent we present in appendix the some useful formula for gamma matrices and Majorana spinors.

2 Notations and preliminary results

To define our notations we list the features of octonion algebra and some other mathematical structures as far as they are of relevance to our work. In addition, we prove some simple assertions concerning isomorphisms and automorphisms of the octonion algebra.

2.1 Algebra of octonions

We recall that the algebra \mathbb{O} of octonions is a real linear algebra with the canonical basis $1, e_1, \dots, e_7$ such that

$$e_i e_j = -\delta_{ij} + c_{ijk} e_k, \quad (2.1)$$

where the structure constants c_{ijk} are completely antisymmetric and nonzero and equal to unity for the seven combinations (or cycles)

$$(ijk) = (123), (145), (167), (246), (275), (374), (365).$$

The algebra of octonions is not associative but alternative, i.e. the associator

$$(x, y, z) = (xy)z - x(yz) \quad (2.2)$$

is totally antisymmetric in x, y, z . Consequently, any two elements of \mathbb{O} generate an associative subalgebra. The algebra \mathbb{O} permits the involution (antiautomorphism of period two) $x \rightarrow \bar{x}$ such that the elements

$$t(x) = x + \bar{x} \quad \text{and} \quad n(x) = \bar{x}x \quad (2.3)$$

are in \mathbb{R} . In the canonical basis this involution is defined by $\bar{e}_i = -e_i$. It follows that the bilinear form

$$\langle x, y \rangle = \frac{1}{2}(\bar{x}y + \bar{y}x) \quad (2.4)$$

is positive definite and defines an inner product on \mathbb{O} . Obviously, it is invariant under all automorphisms of \mathbb{O} . It is easy to prove that the quadratic form $n(x)$ is positive definite and permits the composition

$$n(xy) = n(x)n(y). \quad (2.5)$$

Since the quadratic form $n(x)$ is positive definite, it follows immediately from (2.5) that \mathbb{O} is a division algebra.

There is an explicit procedure for building the algebra of octonions. Suppose e is an element in \mathbb{O} such that $\bar{e} = -e$ and $n(e) = 1$. We choose a quaternion subalgebra \mathbb{H} so that $\mathbb{H} \perp e$ and define a multiplication on the vector space direct sum $\mathbb{H} \oplus \mathbb{H}e$ by

$$(x_1 + y_1e)(x_2 + y_2e) = (x_1x_2 - \bar{y}_2y_1) + (y_2x_1 + y_1\bar{x}_2)e. \quad (2.6)$$

Obviously, $\mathbb{H}e$ is an orthogonal complement to \mathbb{H} relative to the form (2.4). We denote this space by the symbol \mathbb{H}^\perp . It can easily be checked that the algebra $\mathbb{H} \oplus \mathbb{H}^\perp$ with the multiplication (2.6) is the algebra of octonions. Note also that \mathbb{O} is unique, to within an isomorphism, nonassociative composition division algebra. We refer for proof to Ref. 7.

2.2 Malcev algebras and Moufang loops

Since the algebra of octonions is nonassociative, its commutator algebra $\mathbb{O}^{(-)}$ is non-Lie. Instead of the Jacobi identity the algebra $\mathbb{O}^{(-)}$ satisfies the Malcev identity

$$J(x, y, [x, z]) = [J(x, y, z), x], \quad (2.7)$$

where

$$J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y] \quad (2.8)$$

is so-called Jacobian of x, y, z . We define the seven-dimensional Malcev sub-algebra

$$\mathbb{M} = \{x \in \mathbb{O}^{(-)} \mid t(x) = 0\}. \quad (2.9)$$

Obviously, the algebra \mathbb{M} has the basis e_1, \dots, e_7 . Using (2.1) we can find the commutators and Jacobians of the basis elements

$$[e_i, e_j] = 2c_{ijk}e_k, \quad (2.10)$$

$$J(e_i, e_j, e_k) = 12c_{ijkl}e_l. \quad (2.11)$$

Here c_{ijkl} is a completely antisymmetric nonzero tensor equal to unity for the seven combinations

$$(ijkl) = (4567), (2367), (2345), (1357), (1364), (1265), (1274).$$

An anticommutative algebra satisfying the identity (2.7) is called a Malcev algebra⁸. The Malcev algebra (2.9) has a particular importance. It is known⁹ that any real compact simple non-Lie Malcev algebra is isomorphic to the algebra \mathbb{M} . In addition, any semisimple Malcev algebra of characteristic 0 is decomposed in a direct sum of simple algebras. In particular, any semisimple Malcev algebra is isomorphic to a subalgebra of commutator algebra for some alternative algebra.

Recall¹⁰, that a loop is a binary system with a unity element, in which the equations $ax = b$ and $ya = b$ are uniquely solvable. An analytic loop is an analytic manifold equipped with the loop structure, in which the binary operations are analytic. Since the algebra of octonions \mathbb{O} is a real division algebra, the set \mathbb{O}^* of all nonzero elements of \mathbb{O} is an analytic loop. It is easy to prove that \mathbb{O}^* satisfies the identities

$$(xy)(zx) = x((yz)x), \quad ((zy)z)x = z(y(zx)), \quad x(y(zy)) = ((xy)z)y \quad (2.12)$$

which are called the central, left, and right Moufang identities accordingly. In any loop two of them are a corollary of third. A loop is called a Moufang loop if it satisfies the identities (2.12). Note that by (2.5) the set

$$\mathbb{S} = \{x \in \mathbb{O}^* \mid n(x) = 1\} \quad (2.13)$$

is closed relative to the multiplication defined by (2.6). Consequently, \mathbb{S} is an analytic Moufang loop. It is known¹¹ that \mathbb{S} is unique, to within an

isomorphism, analytic compact simple nonassociative Moufang loop and its tangent algebra is isomorphic to the Malcev algebra \mathbb{M} . In addition, any semisimple analytic Moufang loop is decomposed in a direct product of simple Moufang loops. Everywhere below we denote the algebra (2.9) and the loop (2.13) by the symbols \mathbb{M} and \mathbb{S} respectively.

2.3 Isomorphisms and automorphisms

We will use the following construction. Let u be a fixed element of \mathbb{S} . We define a new multiplication in \mathbb{O} by

$$x \circ y = (xu^{-3})(u^3y). \quad (2.14)$$

Obviously, the multiplication (2.14) converts the vector space \mathbb{O} into a linear algebra. We denote this algebra by the symbol \mathbb{O}' . It is easy to prove the following.

Proposition 1. *The algebras \mathbb{O} and \mathbb{O}' are isomorphic.*

Proof. In the first place, note that the algebra \mathbb{O}' is composition. Indeed, the quadratic form $n(x) = \bar{x}x$ is defined on the space \mathbb{O}' . Using (2.5), we prove the identity $n(x \circ y) = n(x)n(y)$. Secondly, the equations $a \circ x = b$ and $y \circ a = b$ are uniquely solvable in \mathbb{O}' . In the third place, the dimensions of \mathbb{O} and \mathbb{O}' are coincided. Thus, \mathbb{O}' is an eight-dimensional composition division algebra. Using the classification of composition algebras, we prove the isomorphism $\mathbb{O}' \simeq \mathbb{O}$.

We construct the isomorphism $\mathbb{O} \rightarrow \mathbb{O}'$ in the explicit form. Supposed \mathbb{H} is a quaternion subalgebra in \mathbb{O} such that $u \in \mathbb{H}$. We consider the mapping $\alpha : \mathbb{O} \rightarrow \mathbb{O}'$ such that

$$\begin{aligned} \alpha(x) &= x, & \text{if } x \in \mathbb{H}, \\ \alpha(x) &= u^{-3}x, & \text{if } x \in \mathbb{H}^\perp. \end{aligned} \quad (2.15)$$

Proposition 2. *The mapping $\alpha : \mathbb{O} \rightarrow \mathbb{O}'$ defined by (2.15) is an isomorphism of the algebras.*

Proof. Denote by x' the element $\alpha(x)$. Using (2.6) we proof by direct calcu-

lation that

$$\begin{aligned}(xy)' &= x' \circ y', \\ (x(ye))' &= x' \circ (ye)', \\ ((ye)x)' &= (ye)' \circ x', \\ ((xe)(ye))' &= (xe)' \circ (ye)'\end{aligned}$$

for any $x, y \in \mathbb{H}$ and $e \in \mathbb{H}^\perp$ with $n(e) = 1$. Consequently, the mapping $\alpha : \mathbb{O} \rightarrow \mathbb{O}'$ is an isomorphism.

The equalities (2.15) define not only the isomorphism $\mathbb{O} \rightarrow \mathbb{O}'$ of the algebras but also a linear transformation of the space \mathbb{O} . Suppose

$$\beta(x) = uxu^{-1}, \quad (2.16)$$

$$\varphi(x) = ux'u^{-1}. \quad (2.17)$$

[Here and everywhere below we denote the element $\alpha(x)$ by the symbols x' .] Then we have the following.

Proposition 3. *The linear transformation $\varphi = \beta\alpha$, defined by (2.17) is an automorphism of \mathbb{O} .*

Proof. On the one hand,

$$\begin{aligned}\varphi(xy) &= u(x' \circ y')u^{-1}, \\ \varphi(x)\varphi(y) &= (ux'u^{-1})(uy'u^{-1}).\end{aligned} \quad (2.18)$$

On the other hand, it follows from the Moufang identities (2.12) that

$$x^{-1}((xy)z) = (yx^{-1})(xz) = (y(zx^{-1}))x. \quad (2.19)$$

Using alternativity of \mathbb{O} and the identities (2.19), we get

$$u(x \circ y)u^{-1} = (uxu^{-1})(uyu^{-1}). \quad (2.20)$$

Comparing (2.18) and (2.20), we prove the proposition.

Obviously, the mapping (2.15) is not defined only by selection of u . We must also fix a quaternion subalgebra of \mathbb{O} containing this element. To this end we fix an element ψ in \mathbb{M} . It is obvious that the couple (u, ψ) generates a quaternion subalgebra if $u\psi u^{-1} \neq \psi$. In this case we say that (u, ψ) defines the transformations (2.15) and (2.17).

Proposition 4. *Let ψ be a fix element of \mathbb{M} . Then the transformations (2.17) defined by (u, ψ) for all $u \in \mathbb{S}$ generate the groups $\text{Aut } \mathbb{O}$ of all automorphisms of \mathbb{O} .*

Proof. First note that all such transformations generate a subgroup of $\text{Aut } \mathbb{O}$. Further, let x be a nonzero element in \mathbb{M} and \mathbb{H} be a quaternion subalgebra of \mathbb{O} containing x and ψ . Then for any u in \mathbb{H} such that $uxu^{-1} \neq x$ the transformation defined by (u, ψ) does not leave fixed x . On the other hand, the group $\text{Aut } \mathbb{O}$ is isomorphic to G_2 . Therefore any maximum subgroup of $\text{Aut } \mathbb{O}$ is isomorphic either to $SU(3)$ or $SO(4)$. If we observe that these subgroups leave fixed the elements of \mathbb{M} , we prove the proposition.

3 Nonassociative gauge theory

In this section we construct a nonassociative gauge theory. At first we give a brief summary of representation theory of Malcev algebras. Then we introduce gauge fields taking their values in the algebra \mathbb{M} and find a transformation law of these fields under the gauge transformations. Further, we construct a field strength tensor and find its transformation law under these transformations. In the end of the section we show that our theory admits the Hamilton gauge.

3.1 Representations of Malcev algebras

Let M be a finite-dimensional semisimple Malcev algebra over a field F of characteristic 0. Without loss of generality it can be assumed that the algebra M is embedded in a commutator algebra of alternative algebra. Suppose V is a vector space over F and $T : M \rightarrow \text{End } V$ ($x \mapsto T_x$) is a linear mapping. Then T is called a representation of M if this algebra defined on the direct sum $M \oplus V$ by means of

$$[v + x, w + y] = T_x w - T_y v + [x, y] \quad (3.1)$$

is a Malcev algebra. In this case V is said to be a Malcev module for M or M module. It follows from (2.7) that the operators T_x satisfy

$$T_{[[x,y],z]} = T_z T_y T_x - T_y T_x T_z + T_x T_{[y,z]} - T_{[z,x]} T_y. \quad (3.2)$$

Conversely, if for all $x, y, z \in M$ the equation (3.2) is true, then T is a representation of M .

A special case of the representation is the mapping $T : M \rightarrow \text{End } M$ that defined by the equations

$$T_x y = [x, y] \quad (3.3)$$

for all $y \in M$. This representation is said to be regular (or adjoint). Second example of the representation comes out if we consider the mapping $T : M \rightarrow \text{End } V$ satisfying

$$T_{[x,y]} = [T_x, T_y] \quad (3.4)$$

for all $x, y \in M$. Since (3.2) is a corollary of (3.4), this mapping is really representations of M (and a homomorphism of M into a Lie algebra of linear transformations of V). Such representations are important for the theory of Lie algebras; however, their significance is not too large in the theory of Malcev algebras.

Nevertheless, the representation theory of Malcev algebras is analogous to the representation theory of Lie algebras. It is known⁹ that any representation of a semisimple Malcev algebra is completely reducible. Any irreducible Malcev module is either Lie or the regular module for a nonassociative simple Malcev algebra or $sl(2)$ module of dimension 2 such that $T_x = x^*$, where x^* is the adjoint matrix to $x \in sl(2)$. Note also that the representation theory can be extend to Moufang loops¹².

The situation is very simple if we have the algebra \mathbb{M} . Any nontrivial representation of \mathbb{M} is regular; the operators T_x are defined by (3.3) and generate the Lie algebra $so(7)$. The latter is decomposed into the direct sum $D(\mathbb{M}) \oplus T(\mathbb{M})$ of the algebra $D(\mathbb{M})$ of derivations of \mathbb{M} and the seven-dimensional subspace $T(\mathbb{M})$. In addition, the Lie brackets are given by

$$[T_x, T_y] = D_{x,y} - T_{[x,y]}, \quad (3.5)$$

$$[D_{x,y}, T_z] = T_{D_{x,y}z}, \quad (3.6)$$

where $D_{x,y}$ is an operator of derivations of \mathbb{M} . It is well known that the algebra $D(\mathbb{M})$ is isomorphic to the exceptional Lie algebra g_2 . Obviously, the algebras of derivations of \mathbb{M} and \mathbb{O} are coincided.

3.2 Gauge transformations

We will now apply the representation theory of Malcev algebras to a construction of gauge theory. Let $A_\mu(x)$ be a vector field taking its value in

\mathbb{M} and $\psi(x)$ be a field taking its value in a space V of representation of \mathbb{M} . Denote by \hat{A}_μ the operator T_{A_μ} and define the covariant derivative

$$D_\mu \psi = \partial_\mu \psi + \hat{A}_\mu \psi. \quad (3.7)$$

Obviously, the spaces V and \mathbb{M} are coincided and the operator \hat{A}_μ is defined by

$$\hat{A}_\mu \psi = [A_\mu, \psi]. \quad (3.8)$$

As in the Yang-Mills theory, the gauge field is endowed with a transformation law under gauge transformations such that $D_\mu \psi$ transform as ψ , i.e.,

$$\psi \rightarrow U\psi, \quad (3.9)$$

$$D_\mu \psi \rightarrow U(D_\mu \psi), \quad (3.10)$$

where $U = U(x)$ is a function taking its values in the group $\text{Aut } M$ of all automorphisms of \mathbb{M} .

We will now find a transformation law of A_μ under the gauge transformations (3.9). From (3.9) and (3.10), we get the usual transformation law of operator functions

$$\partial_\mu + \hat{A}_\mu \rightarrow \partial_\mu + U\hat{A}_\mu U^{-1} + U\partial_\mu U^{-1}. \quad (3.11)$$

Since $\hat{A}_\mu = T_{A_\mu}$ and $U \in \text{Aut } M$, we have

$$U\hat{A}_\mu U^{-1} = T_{U A_\mu}. \quad (3.12)$$

On the other hand, it follows from Propositions 3 and 4 that the function $U(x)$ can be chosen as the composition $U = \beta\alpha$ of transformations defined by (2.15) and (2.16). By Proposition 2, it follows that the operator function $\alpha(x)$ defines the isomorphism $\mathbb{O} \rightarrow \mathbb{O}'$ for any value of x . Suppose $\psi' = \alpha(\psi)$ and define its derivation by

$$\nabla_\mu \psi' = (\partial_\mu \psi)'. \quad (3.13)$$

It is easy to prove that any two differentiable functions $f(x)$ and $g(x)$ taking their values in \mathbb{O}' satisfy

$$\nabla_\mu (f \circ g) = \nabla_\mu f \circ g + f \circ \nabla_\mu g. \quad (3.14)$$

Noting that the operator

$$\nabla_\mu = \partial_\mu + \alpha \partial_\mu \alpha^{-1}, \quad (3.15)$$

and using (3.12) and (3.14), we get

$$\partial_\mu + T_{A_\mu} \rightarrow \nabla_\mu + T_{UA_\mu} + \beta \nabla_\mu \beta^{-1} \quad (3.16)$$

instead of (3.11).

Suppose that the transformations (3.9) and (3.10) are infinitesimal. Then the operator functions α and β take the form

$$\alpha(x) = 1 + \Gamma(x) \quad (3.17)$$

$$\beta(x) = 1 + T_{\theta(x)}, \quad (3.18)$$

where $\theta(x)$ is defined by $u(x) = 1 + \theta(x)$ that takes its value in a neighborhood of unity element of \mathbb{S} . In this case we can consider the transformations

$$\partial_\mu \rightarrow \nabla_\mu = \partial_\mu - \partial_\mu \Gamma, \quad (3.19)$$

$$A_\mu \rightarrow A'_\mu + [\theta, A_\mu] - \partial_\mu \theta, \quad (3.20)$$

where $A'_\mu = A_\mu + \Gamma A_\mu$, instead of (3.16). The formula (3.20) gives us a transformation law of A_μ under the gauge transformations (3.9). Notice that in contrast with the Yang-Mills theory, we have the transformation (3.19). As usual, we define a finite gauge transformation as an infinite sequence of infinitesimal transformations.

We now want to construct the field strength tensor in the nonassociative case. Denote by $\hat{F}_{\mu\nu}$ a projection of the Lie bracket $[D_\mu, D_\nu]$ onto $T(\mathbb{M})$. Using (3.5), we get

$$\hat{F}_{\mu\nu} = T_{F_{\mu\nu}}, \quad (3.21)$$

where the tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \quad (3.22)$$

Since the subspace $T(\mathbb{M})$ is G_2 invariant, it follows that (3.11) induces the transformation

$$\hat{F}_{\mu\nu} \rightarrow U \hat{F}_{\mu\nu} U^{-1}. \quad (3.23)$$

Using (3.12) and (3.21), we get the transformation law

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} = F'_{\mu\nu} + [\theta, F_{\mu\nu}] \quad (3.24)$$

of the tensor (3.22) under the infinitesimal gauge transformations (3.19) and (3.20). It follows from (3.24) that the field strength tensor may be really defined by (3.22). Notice that the tensor $F_{\mu\nu}$ takes more habitual form in the basis $\tilde{e}_i = -e_i$. In this basis $\tilde{A}_\mu = -A_\mu$ and $\tilde{F}_{\mu\nu} = -F_{\mu\nu}$.

3.3 Hamilton gauge

In the Yang-Mills theory, owing to the gauge arbitrariness, we may demand that the potential locally satisfies a definite condition. The situation is similar in the nonassociative case. There exists a gauge transformation $A_\mu \rightarrow A_\mu^\varphi$ such that

$$A_0^\varphi(x) = 0. \quad (3.25)$$

Indeed, the potential $A_0(\mathbf{x}, t) \rightarrow 0$ as $t \rightarrow -\infty$. Therefore there exists t_1 such that the equation

$$\frac{\partial u}{\partial t} = u A_0, \quad (3.26)$$

where $u(x)$ takes its values in a neighborhood of unity element of \mathbb{S} , has the solution

$$u_1(\mathbf{x}, t) = 1 + \int_{-\infty}^t A_0(\mathbf{x}, s) ds$$

for all $t \in [-\infty, t_1]$. Since the mapping $\mathbb{O} \rightarrow \mathbb{O}'$ defined by (2.15) is isomorphism, it follows from (3.26) that the function

$$A_{0,1} = u_1(A'_0 + \nabla_0)u_1^{-1}$$

satisfies $A_{0,1} = 0$ on this interval. It is clear that A_0 and $A_{0,1}$ are connected by an infinitesimal gauge transformation.

Further, let $t_{n+1} = t_n + \delta t_n$, where $n \in \mathbb{N}$. It is readily seen that the equation

$$\frac{\partial u}{\partial t} = u A_{0,n} \quad (3.27)$$

has the solution

$$u_{n+1}(\mathbf{x}, t) = 1 + \int_{-\infty}^t A_{0,n}(\mathbf{x}, s) ds$$

on the interval $[-\infty, t_{n+1}]$ if the function $A_{0,n}(x)$ is defined by

$$A_{0,n} = u_n (A'_{0,n-1} + \nabla_0) u_n^{-1}$$

with $A_{0,0} = A_0$. From (3.27) it follows that $A_{0,n+1} = 0$ for all $t < t_{n+1}$. If we suppose

$$A_0^\varphi(x) = \lim_{n \rightarrow \infty} A_{0,n}(x),$$

and use the induction on n , we prove (3.25). The functions $A_0(x)$ and $A_0^\varphi(x)$ are connected by a gauge transformation. Hence in every class of gauge-equivalent fields, there exists a field satisfying the condition (3.25).

4 Supersymmetric gauge theory

In this section we construct an on-shell version of $N = 1$ supersymmetric gauge theory without matter. The model is described by a vector field A_μ and by a Majorana spinor field ψ . All fields take their values in the Malcev algebra \mathbb{M} .

4.1 Supersymmetry transformations

We examine the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu} \rangle + \frac{i}{2}\langle \bar{\psi}, \gamma^\mu D_\mu \psi \rangle. \quad (4.1)$$

It contains the covariant derivative $D_\mu \psi$ and the field strength tensor $F_{\mu\nu}$ defined by (3.7) and (3.22), respectively. Since the inner product (2.4) is invariant under all automorphisms of \mathbb{O} , it follows from (3.9), (3.10) and (3.24) that the Lagrangian density (4.1) is invariant under the gauge transformations (3.19) and (3.20).

We will prove that the action with this Lagrangian density is invariant under the following supersymmetry transformations:

$$\delta A_\mu = i\bar{\varepsilon}\gamma_\mu\psi, \quad (4.2)$$

$$\delta\psi = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\varepsilon, \quad (4.3)$$

where ε is a constant anticommuting Majorana spinor. To calculate the variation of the Lagrangian density one needs the formulas (A.2), (A.3), (A.6) and (A.8) in the Appendix. Using these formulas and the identities

$$\begin{aligned} \delta F_{\mu\nu} &= i\bar{\varepsilon}(\gamma_\nu D_\mu - \gamma_\mu D_\nu)\psi, \\ \delta(D_\mu\psi) &= D_\mu\delta\psi + [\delta A_\mu, \psi], \end{aligned}$$

we get

$$\delta\mathcal{L} = \frac{1}{2}\langle \bar{\psi}\gamma^\mu, [\psi, \bar{\varepsilon}\gamma_\mu\psi] \rangle + \frac{i}{2}\langle D_\rho F_{\mu\nu}, \bar{\varepsilon}\gamma^{\mu\nu\rho}\psi \rangle - \frac{1}{2}\bar{\varepsilon}\partial_\mu V^\mu, \quad (4.4)$$

with

$$V^\mu = \langle iF^{\mu\nu}, \gamma_\nu\psi \rangle + \langle {}^*F^{\mu\nu}, \gamma_5\gamma_\nu\psi \rangle,$$

where $*F^{\mu\nu}$ is a dual field strength tensor.

It is easy to prove that the first term in the right hand side of (4.4) vanishes. Indeed, the tensor c_{ijk} defined by (2.1) is completely antisymmetric. Therefore we can act as in the supersymmetric Yang-Mills theory. We rewrite this term as

$$\frac{1}{2}\langle\bar{\psi}\gamma^\mu, [\psi, \bar{\varepsilon}\gamma_\mu\psi]\rangle = c_{ijk}(\bar{\varepsilon}\gamma_\mu\psi^k)(\bar{\psi}^i\gamma^\mu\psi^j). \quad (4.5)$$

Then we insert the Fierz identity (A.5) for $\psi^k\bar{\psi}^i$ in the right hand side of (4.5) and use the relations (A.4) in the appendix. We get

$$\begin{aligned} (\bar{\psi}^i\gamma^\mu\psi^j)(\bar{\varepsilon}\gamma_\mu\psi^k) &= -(\bar{\varepsilon}\psi^j)(\bar{\psi}^i\psi^k) + \frac{1}{2}(\bar{\varepsilon}\gamma_\mu\psi^j)(\bar{\psi}^i\gamma^\mu\psi^k) \\ &\quad - \frac{1}{2}(\bar{\varepsilon}\gamma_\mu\gamma_5\psi^j)(\bar{\psi}^i\gamma_5\gamma^\mu\psi^k) + (\bar{\varepsilon}\gamma_5\psi^k)(\bar{\psi}^i\gamma_5\psi^j), \end{aligned}$$

where all but the second term on the right hand side is symmetric in i and k . Using this identity, we prove that the expression on the left in (4.5) is zero.

We now examine the second term in the right hand side of (4.4). Since the algebra \mathbb{M} is non-Lie, the tensor $*F_{\mu\nu}$ does not satisfy the Bianchi identity. Therefore it is not obvious that this term is zero. Let η be a constant anticommuting Majorana spinor such that $\bar{\eta}\eta = 1$, and let $\varepsilon = a\eta$ for $a \in \mathbb{R}$. Using the identities (A.9) and (A.10) in the Appendix, we get

$$(\bar{\eta}\gamma_5\eta)(\bar{\varepsilon}\gamma_\mu\psi) = \bar{\psi}\gamma_\mu\gamma_5\varepsilon. \quad (4.6)$$

Using (4.6) and (A.3), we get

$$k\varepsilon^{\mu\nu\rho\sigma}(\bar{\varepsilon}\gamma_\sigma\psi) = \bar{\varepsilon}\gamma^{\mu\nu\rho}\psi, \quad (4.7)$$

where $k = i\bar{\eta}\gamma_5\eta$. It follows from (4.7) that

$$i\langle D_\rho F_{\mu\nu}, \bar{\varepsilon}\gamma^{\mu\nu\rho}\psi \rangle = k\varepsilon^{\mu\nu\rho\sigma}\langle D_\rho F_{\mu\nu}, i\bar{\varepsilon}\gamma_\sigma\psi \rangle. \quad (4.8)$$

On the other hand, it is easy to prove that

$$3\varepsilon^{\mu\nu\rho\sigma}D_\rho F_{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}J(A_\mu, A_\nu, A_\rho), \quad (4.9)$$

where the Jacobian $J(A_\mu, A_\nu, A_\rho)$ is defined by (2.8). Using (2.11) and (4.2), we get

$$\varepsilon^{\mu\nu\rho\sigma}\langle D_\rho F_{\mu\nu}, i\bar{\varepsilon}\gamma_\sigma\psi \rangle = \varepsilon^{\mu\nu\rho\sigma}c_{ijkl}\delta(A_\mu^i A_\nu^j A_\rho^k A_\sigma^l). \quad (4.10)$$

In the Hamilton gauge the right hand side of (4.10) vanishes. Since the action with the Lagrangian density defined in (4.1) is gauge invariant, we conclude that the second term in (4.4) is absent and that the supersymmetric variation of the Lagrangian density is just a divergence.

4.2 Superalgebra

A basic algebraic fact about supersymmetry is that the commutator of two supersymmetry transformations gives a spatial translation. This is true for our theory. Indeed, using the formulas (A.1), (A.2), (A.6), and (A.7) in the Appendix and the obvious identity $\gamma_{\mu\nu}\gamma^\nu = 3\gamma_\mu$, we prove that on shell the commutators

$$[\delta_1, \delta_2]A_\mu = -2i(\bar{\varepsilon}_2\gamma^\nu\varepsilon_1)\partial_\nu A_\mu + D_\mu\theta, \quad (4.11)$$

$$[\delta_1, \delta_2]\psi = -2i(\bar{\varepsilon}_2\gamma^\nu\varepsilon_1)\partial_\nu\psi + [\psi, \theta]. \quad (4.12)$$

The gauge parameter $\theta = 2i(\bar{\varepsilon}_2\gamma^\nu A_\nu\varepsilon_1)$ depends on the gauge field and the supersymmetric parameters ε_i . Here we use the fact that ψ obeys the Dirac equation $\gamma^\mu D_\mu\psi = 0$. Further, we consider the consequence

$$(A_\mu, \psi) \xrightarrow{U} (\tilde{A}_\mu, \tilde{\psi}) \xrightarrow{\Phi} (\tilde{\tilde{A}}_\mu, \tilde{\tilde{\psi}})$$

of two gauge transformations U and Φ . Here U is an infinitesimal transformation and Φ is a finite transformation. It follows from (3.9) and (3.20) that the transformation U is

$$\begin{aligned} \delta\psi &= \Gamma\psi + [\theta, \psi], \\ \delta A_\mu &= \Gamma A_\mu + [\theta, A_\mu] - \partial_\mu\theta. \end{aligned}$$

On the other hand, it follows from (3.19) that the transformation Φ defines the mapping $\partial_\mu \rightarrow \nabla_\mu$, where the covariant operator ∇_μ is given by (3.15). If we choose the infinitesimal function

$$\Gamma = -2i(\bar{\varepsilon}_2\gamma^\nu\varepsilon_1)\alpha\partial_\mu\alpha^{-1},$$

then from (4.11) and (4.12) we get the operator relation

$$[\delta_1, \delta_2] = -2i(\bar{\varepsilon}_2\gamma^\nu\varepsilon_1)\partial_\nu. \quad (4.13)$$

Thus, as in the supersymmetric Yang-Mills theory this superalgebra closes only on gauge invariant fields.

4.3 Chiral representation

In spite of the fact that in the simplest $N = 1$ supersymmetric gauge theories one usually uses Majorana spinors, it is very desirable to examine our pattern

in the chiral representation. Primarily, we rewrite the Lagrangian density and the supersymmetry transformations of the theory in terms of Weyl spinors. Suppose

$$\mathcal{L}' = -\frac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu} \rangle + \frac{i}{2}\langle \bar{\psi}, \gamma^\mu D_\mu \psi \rangle + \frac{k}{4}\langle F_{\mu\nu}, {}^*F^{\mu\nu} \rangle, \quad (4.14)$$

where ψ is a left-handed spinor and k is a constant. It is obvious that \mathcal{L}' is invariant under the gauge transformations (3.19) and (3.20). We consider the following supersymmetry transformations:

$$\delta A_\mu = \frac{i}{2} \{ \bar{\varepsilon} \gamma_\mu \psi - (\bar{\varepsilon} \gamma_\mu \psi)^\dagger \}, \quad (4.15)$$

$$\delta \psi = \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \varepsilon, \quad (4.16)$$

where ε is a constant anticommuting left-handed Weyl spinor. As above, we calculate the variation

$$\delta \mathcal{L}' = \frac{i}{2} \langle D_\rho F_{\mu\nu}, (\bar{\varepsilon} \gamma^{\mu\nu\rho} \psi - k \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_\sigma \psi) \rangle - \frac{1}{2} \bar{\varepsilon} \partial_\mu \tilde{V}^\mu + H.c. \quad (4.17)$$

Using the formula $\gamma_5 \psi = \psi$ and the identities (A.3) in the Appendix, we prove that the first term in the right hand side of (4.17) vanishes only if $k = i$. Thus, the last term in the right hand side of (4.14) is purely imaginary. Arguing as the end of Subsection 4.1, we see that in four dimensions the last term in the right hand side of (4.14) is a divergence though. Consequently, the action with the Lagrangian density defined in (4.14) is invariant under the supersymmetry transformations (4.15) and (4.16).

5 Conclusion

In this paper we have given a construction of nonassociative gauge theory in which the Moufang loop is used instead of the structure group. We have also demonstrated how this theory can be used to construct an on-shell version of $N = 1$ supersymmetric gauge theory without matter.

In contrast to the Yang-Mills theory, we have studied not only transformations of the gauge field but also transformations of the operator of differentiation. This is a characteristic feature of the gauge theory. Because of this we may demand that the potential locally satisfies a definite condition. In particular, we may choose the Hamilton gauge. It is obvious that

the gauge theory can be defined in spaces of dimension greater than 4. In addition, it can be easily generalized if we take a real semisimple Malcev algebra instead of the algebra \mathbb{M} . Since any Lie algebra is Malcev, it follows that such gauge theory is a generalization of the Yang-Mills theory.

Conversely, it is not clear how the $N = 1$ supersymmetric gauge theory can be defined in spaces of dimension greater than 4 and how the simplest four-dimensional supersymmetric theory can be extended to theories with extended supersymmetry. In addition, there is the challenge to couple the $N = 1$ supersymmetric gauge theory to the supergravity system so that the combined system is invariant under the local supersymmetric transformations. It is unsatisfying to be limited to the simple example of supersymmetric gauge theory that we have considered without evidence that more general possibilities are not viable. Therefore there are a lot of open problems, which deserve further study.

Acknowledgements

The research was supported by RFBR Grant No. 06-02-16140.

A Appendix

In this appendix we collected some useful formulas which are used in the main body of the paper. In particular, we use the commutation relations

$$[\gamma_{\mu\nu}, \gamma_{\rho\sigma}] = 2(\gamma_{\mu\sigma}g_{\nu\rho} + \gamma_{\nu\rho}g_{\mu\sigma} - \gamma_{\mu\rho}g_{\nu\sigma} - \gamma_{\nu\sigma}g_{\mu\rho}), \quad (\text{A.1})$$

of gamma matrices in four dimensions and the products

$$\begin{aligned} \gamma_{\mu\nu}\gamma_{\rho} &= \gamma_{\mu\nu\rho} + g_{\nu\rho}\gamma_{\mu} - g_{\mu\rho}\gamma_{\nu}, \\ \gamma_{\rho}\gamma_{\mu\nu} &= \gamma_{\mu\nu\rho} - g_{\nu\rho}\gamma_{\mu} + g_{\mu\rho}\gamma_{\nu}, \end{aligned} \quad (\text{A.2})$$

where we use the notation $\gamma_{\mu\nu\dots}$ for a totally antisymmetrized product of $\gamma_{\mu}\gamma_{\nu}\dots$. We also use the simple relations

$$\begin{aligned} \gamma^{\mu\nu\rho\sigma}\gamma_{\sigma} &= \gamma^{\mu\nu\rho}, \\ \varepsilon^{\mu\nu\rho\sigma}\gamma_5 &= i\gamma^{\mu\nu\rho\sigma}, \end{aligned} \quad (\text{A.3})$$

where $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$, and the conjugation formulas

$$\begin{aligned} \gamma_{\mu}\gamma_{\rho}\gamma^{\mu} &= -2\gamma_{\rho}, \\ \gamma_{\mu}\gamma_{\rho\sigma}\gamma^{\mu} &= 0, \\ \gamma_{\mu}\gamma_5\gamma_{\rho}\gamma^{\mu} &= -2\gamma_{\rho}\gamma_5. \end{aligned} \quad (\text{A.4})$$

We take two spinor ψ and χ whose components anticommute. The Fierz identity

$$4\psi\bar{\chi} = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) + \frac{1}{2}\gamma_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar{\chi}\gamma_5\gamma^\mu\psi) - \gamma_5(\bar{\chi}\gamma_5\psi) \quad (\text{A.5})$$

allows us to write the matrix $\psi\bar{\chi}$ as a linear combination of the antisymmetrized products of γ matrices. The following identities follow from the identity (A.5):

$$\bar{\psi}\gamma^\rho\gamma_{\mu\nu}\chi = \bar{\chi}\gamma_{\mu\nu}\gamma^\rho\psi, \quad (\text{A.6})$$

$$\chi\bar{\psi} - \psi\bar{\chi} = -\frac{1}{2}\gamma_\mu(\bar{\psi}\gamma^\mu\chi) + \gamma_{\mu\nu}(\bar{\psi}\gamma^{\mu\nu}\chi). \quad (\text{A.7})$$

We define the hermitian conjugate as if the spinor components are operators in a Hilbert space. For Majorana spinors we have

$$(\bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi)^\dagger = (-1)^n\bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi, \quad (\text{A.8})$$

$$\bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi = (-1)^{n(n-1)/2}\bar{\chi}\gamma_{\mu_1\dots\mu_n}\psi. \quad (\text{A.9})$$

In particular, $\bar{\psi}\gamma_\mu\psi = \bar{\psi}\gamma_{\mu\nu}\psi = 0$. Setting $\chi = \psi$ in (A.5) we obtain for Majorana spinors the identity

$$\psi(\bar{\psi}\psi) + \gamma_5\psi(\bar{\psi}\gamma_5\psi) = 0. \quad (\text{A.10})$$

In the paper we use a Majorana representation of the Dirac algebra in which the gamma matrices are all imaginary and the spinors are real.

References

- [1] F. Gursey and C. H. Tze, *On the role of division, Jordan and related algebras in particle physics* (World Scientific, Singapore, 1996).
- [2] M.J. Duff, D.E. Nilsson, and C.N. Pope, Phys. Rept. **130**, 1 (1986).
- [3] J.A. Harvey and A. Strominger, Phys. Rev. Lett. **66**, 549 (1991);
M.J. Duff, R.R. Khuri, and J.H. Lu, Phys. Rept. **259**, 213 (1995);
B.S. Acharya, M. O’Loughlin, and B. Spence, Nucl. Phys. B **503**, 657 (1997);
I. Bakas, E.G. Floratos, and A. Kehagias, Phys. Lett. B **445**, 69 (1998).
- [4] L. Cornalba and R. Schiappa, Commun. Math. Phys. **225**, 33 (2002).

- [5] A.I. Nesterov, Phys. Lett. A **328**, 110 (2004);
P. de Medeiros and S. Ramgoolam, JHEP **0503**, 072 (2005);
Y. Sasai and N. Sasakura, JHEP **0609**, 046 (2006).
- [6] S. Majid, J. Math. Phys. **46**, 103519 (2005);
M. Gogberashvili, J. Phys. A **39**, 7099 (2006).
- [7] R.D. Schafer, *An Introduction to Non-Associative Algebras* (Academic, New York, 1966).
- [8] A.A. Sagle, Trans. Amer. Math. Soc. **101**, 426 (1961).
- [9] E.N. Kuzmin, Algebra and Logic **7**, 48 (1968) (in Russian).
- [10] R.H. Bruck, *A Survey of Binary System* (Springer, Berlin, 1971).
- [11] E.N. Kuzmin, Algebra and Logic **10**, 3 (1971) (in Russian);
F.S. Kerdman, Algebra and Logic **18**, 523 (1979) (in Russian).
- [12] E.K. Loginov, Nucl. Phys. B **606**, 636 (2001).